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A relativistic mass dipole gravitational theory and its connections with AQUAL

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Abstract

It will be shown that a gravitational theory based on there being an additional contribution to the gravitational field from mass dipoles leads to the same field equation that arises from the AQUAL formulation of MOND. However, unlike AQUAL, the mass dipole theory does not require a modification of Newtonian gravitational theory. In addition, both SR and linearized GR field equations will be derived for the mass dipole and AQUAL gravitational theories.

Keywords: mass dipoles, AQUAL, MOND, dark matter alternative, GRAS, gravitational anti-screening

1. Introduction

MOdified Newtonian Dynamics (MOND) is an alternative to the hypothesis of dark matter (Milgrom 1983a, 1983b, 1983c, Bekenstein and Milgrom 3, Famaey and McGaugh 12, Sanders 40, Banik and Zhao 2). It was proposed as an explanation for the Tully–Fisher relation (Tully and Fisher 43), an empirical relationship between the luminosity of a spiral galaxy and its asymptotic rotation. The relation became even tighter when instead of luminosity the galaxy's total baryonic mass was used (McGaugh *et al* 20). This is referred to as the baryonic Tully Fisher relationship (BTFR). The BTFR, as given by McGaugh (21), is

$$M_{\mathrm{b}} = A\, v_R^4 \tag{1a}$$

with

$$A = (47 \pm 6)\, M_{\odot}\ \mathrm{km}^{-4}\ \mathrm{s}^4, \tag{1b}$$

where v_{r} is the outer rotational velocity of stars in a spiral galaxy with baryonic mass M_{b}. This relationship is a very definitive observational result. Accordingly, the BTFR is the cornerstone

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of alternative theories such as MOND as this relationship does not fall naturally out of the theory of dark matter (Wu and Kroupa 44, Desmond 2017a, 2017b).

Initially MOND involved a modification of the law of inertia which leads to a violation of Newton's 3rd law. AQUAL (A QUADratic Lagrangian), a gravitational theory spawned by MOND, addresses this problem by leaving the law of inertia intact and instead modifies gravity (Bekenstein and Milgrom 3). AQUAL leads to a field equation that reduces to the MOND result in the spherically symmetric case and thereby agrees with the BTFR. AQUAL works well in galaxies but runs into problems when applied to galactic clusters (Sanders 41, McGaugh 22).

Mass dipole gravitational theories are based on there being an additional gravitational contribution from a field of such dipoles surrounding a given baryonic mass. As with MOND and AQUAL, the mass dipole theory leads to agreement with the BTFR. Several mass dipole theories have been presented over the years (Blanchet 2007a, 2007b, Blanchet and Le Tiec 7, Hajdukovic 16, 2011a, 2011b) and more recently GRAS (GRavitational Anti-Screening) (Penner 35, 2016a, 2016b). The GRAS gravitational theory will be discussed in detail in section 3.

In addition to leading to the BTFR, these alternative theories also lead to good agreement with the radial acceleration relationship (RAR). This relationship, following from the work of McGaugh *et al* (23) and Lelli *et al* (18), considers the full rotational curve of spiral galaxies to determine the relationship between their radial acceleration as determined from their rotational curves and the predicted radial acceleration due to their baryonic mass distribution assuming Newtonian gravity to be valid.

The first goal of this manuscript is to show that the gravitational theories of AQUAL and GRAS lead to the same field equation. However, unlike AQUAL, GRAS does not require a modification to Newtonian gravitational theory. The second goal of this manuscript is to provide a relativistic formulation of GRAS and hence indirectly of AQUAL. A relativistic gravitational theory, referred to as TeVeS, has been formulated by Bekenstein (4) which in the weak field limit leads to the MOND result. Although TeVeS has difficulties with regards to both lensing and gravitational waves (Mavromatos *et al* 19, Boran *et al* 8), these can be addressed using the theory of Skordis and Zlosnik (42), especially if postulating a significant sterile neutrino component in galaxy clusters (Haslbauer *et al* 17). The relativistic version of GRAS and AQUAL presented here does not require a modification to GR.

2. AQUAL gravitational theory

In Newtonian gravitational theory, the gravitational field due to a baryonic mass distribution is given by $\mathbf{g_N} = -\nabla\Phi_N$, with the gravitational scalar potential Φ_N determined by Poisson's equation

$$\nabla^2\Phi_N \equiv \nabla \cdot \nabla\Phi_N = 4\pi G\rho_{bo}, \tag{2}$$

where ρ_{bo}, the source of gravity, is the baryonic mass density. The solution to equation (2) is given by

$$\Phi_N = -G\int \frac{\rho_{bo}}{|r - r'|}\ dV'. \tag{3}$$

The Lagrangian corresponding to the Newtonian field equation as given by equation (2) is

$$\mathcal{L} = \frac{1}{8\pi G}|\nabla\Phi|^2 + \rho_{bo}\Phi. \tag{4}$$

In the AQUAL gravitational theory the Lagrangian is given by:

$$\mathcal{L} = \frac{1}{8\pi G} g_o^2 F\left(\frac{|\nabla\Phi|^2}{g_o^2}\right) + \rho_{bo}\Phi, \tag{5}$$

with F being an arbitrary function and g_o being a parameter that is to be determined by observations. This Lagrangian leads to the following AQUAL field equation

$$\nabla\cdot(\mu(|\nabla\Phi|/g_o)\nabla\Phi) = 4\pi G\rho_{bo}, \tag{6a}$$

where the function μ, referred to as the interpolating function, is related to the function F by

$$\mu(x) = F'(x^2). \tag{6b}$$

The interpolating function is such that

$$\mu(g/g_o) \cong 1 \quad \text{for} \ \ g \gg g_o \tag{7a}$$

and

$$\mu(g/g_o) \cong \frac{g}{g_o} \quad \text{for} \ \ g \ll g_o, \tag{7b}$$

with $g = |\nabla\Phi|$. Condition (7a) is required in order to have consistency with Newtonian theory within the Solar System while condition (7b) is required in order to get agreement with the BTFR. Equations (2) and (6a) then lead to the following equation

$$\mu(|\nabla\Phi|/g_o)\nabla\Phi = \nabla\Phi_N, \tag{8a}$$

or equivalently

$$\mu(g/g_o)\mathbf{g} = \mathbf{g}_N. \tag{8b}$$

The interpolating function $\mu(g/g_o)$ therefore determines the deviation between the AQUAL and Newtonian gravitational theories. Many functions will of course fit the conditions given by equations (7a) and (7b). One of these, the standard interpolating function, is given by

$$\mu(g/g_o) = \left(1 + \left(\frac{g_o}{g}\right)^2\right)^{-1/2}. \tag{9}$$

For $g \ll g_o$, by equations (8b) and (9) the relationship between the Newtonian gravitational field g_N and the AQUAL gravitational field g is then

$$g = (g_N g_o)^{1/2}. \tag{10}$$

With the substitutions $g = v^2/r$ and $g_N = GM_b/r^2$, equation (10) leads to the BTFR, as given by equation (1), with $A = (Gg_o)^{-1}$. Of course, the AQUAL field equation was designed to lead to the BTFR and the AQUAL Lagrangian was designed to lead to the AQUAL field equation. Most importantly, with respect to this manuscript, AQUAL is a new gravitational theory which has the same gravitational source as the Newtonian theory.

3. GRAS gravitational theory

GRAS gravitational theory is based on there being an additional contribution to the gravitational field from a field of virtual mass dipoles surrounding a given baryonic mass. It is analogous to the screening of electric charges by virtual electric dipoles that is found in QED. In QED an electric charge is surrounded by a cloud of virtual photons and virtual electric dipoles. These virtual electric dipoles provide a screening effect so that beyond a certain distance the apparent charge of the particle is greatly reduced. In QED it is the observed charge of an electron at $r \gg \lambda_c$ that is equal to value of $-e$. However, as the electron is approached the screening effect provided by the virtual dipoles is reduced thereby resulting in the apparent or bare charge of the electron greatly increasing.

Modelling this effect as a point charge in a dielectric leads to the same behavior as found in QED (Penner 32a). This model was then applied to the hypothetical case where the electric force is such that like charges attract and unlike charges repel. This leads to anti-screening. The behaviour found in this anti-screening case is different than what may be expected. The observed charge does not approach a constant larger value but instead is found to increase with observation distance. This is the same type of behaviour that is found with the BTFR and led to the hypothesis that baryonic masses are surrounded by a field of virtual mass dipoles whose contribution to the gravitational field leads to the BTFR.

Currently the model for GRAS follows from classical theory. In classical electromagnetism, the equivalent charge density ρ_{EM} due to the field of electric dipoles within a dielectric is given by

$$\rho_{EM} = -\nabla \cdot \mathbf{P_{EM}}, \tag{11}$$

$$\mathbf{P_{EM}} = \epsilon_o k \left(E/E_o\right) \mathbf{E}, \tag{12}$$

where $\mathbf{E}$ is the total electric field, $\mathbf{P_{EM}}$ is the electric dipole moment density, and $k\left(E/E_o\right)$ is a function of the electric field magnitude. In the case of a linear dielectric, $k\left(E/E_o\right)$ is equal to the electric susceptibility, otherwise it incorporates any nonlinearity between $\mathbf{P_{EM}}$ and $\mathbf{E}$. Analogous to equation (11) a field of mass dipoles would lead to an equivalent mass density ρ_{do} given by

$$\rho_{do} = -\nabla \cdot \mathbf{P}, \tag{13}$$

where the mass dipole moment density $\mathbf{P}$, is given by

$$\mathbf{P} = N\mathbf{p}, \tag{14}$$

with N being the density of the mass dipoles and $\mathbf{p}$ being the average mass dipole moment of the given particles. Analogously to equation (12), the dependence that the mass dipole moment density $\mathbf{P}$ has on the total gravitational field $\mathbf{g}$ will be given by

$$\mathbf{P} = \frac{1}{4\pi G} f(g/g_o)\mathbf{g}, \tag{15}$$

where the function $f(g/g_o)$ incorporates any nonlinearity between $\mathbf{P}$ and $\mathbf{g}$.

The field equation for GRAS is just that from Newtonian theory (equation (2)) but with an additional mass density ρ_{do} leading to

$$\nabla \cdot \nabla \Phi = 4\pi G \left(\rho_{bo} + \rho_{do}\right), \tag{16a}$$

or by equations (13) and (15)

$$\nabla \cdot \nabla \Phi = 4\pi G \rho_{\mathrm{bo}} + \nabla \cdot (f(|\nabla \Phi|/g_{\mathrm{o}})\nabla \Phi). \tag{16b}$$

Setting

$$f(g/g_{\mathrm{o}}) = 1 - \mu(g/g_{\mathrm{o}}), \tag{17}$$

it is seen that the GRAS and AQUAL field equations are the same. By equation (7), the subsequent conditions on the function $f\left(g/g_{\mathrm{o}}\right)$ are

$$f\left(g/g_{\mathrm{o}}\right) \cong 0 \qquad \text{for } g \gg g_{\mathrm{o}}, \tag{18a}$$

$$f\left(g/g_{\mathrm{o}}\right) \cong 1 - \frac{g}{g_{\mathrm{o}}} \quad \text{for } g \ll g_{\mathrm{o}}. \tag{18b}$$

Just as with AQUAL these condition lead to agreement within the Solar System and with the BTFR. *As such it is proposed that the contribution to the gravitational field by fields of mass dipoles are ultimately responsible for the observed rotational properties of galaxies.*

The Lagrangian for the GRAS gravitational theory is given by

$$\mathcal{L} = \frac{1}{8\pi G}|\nabla \Phi|^2 + \rho_{\mathrm{bo}}\Phi - \frac{1}{8\pi G} g_{\mathrm{o}}^2 K\left(\frac{|\nabla \Phi|^2}{g_{\mathrm{o}}^2}\right), \tag{19}$$

where K is an arbitrary function with $f(x) = K'(x^2)$. This Lagrangian matches that of AQUAL (equation (5)), with the two arbitrary functions related by $K' = 1 - F'$.

Although the GRAS and AQUAL gravitational theories have the same Lagrangians and field equations, the interpretation is different. For AQUAL the only gravitational source is baryonic mass and Newtonian gravitational theory is modified. In the case of GRAS Newtonian gravitational theory is fine but there is an additional gravitational source, namely the field due to the mass dipoles. The mass density ρ_{do} resulting from these dipoles is as real as the baryonic mass density ρ_{bo}, with both contributing to the gravitational field. In some sense, the theory of dark matter and GRAS are similar. With the dark matter hypothesis, it is unknown particles whose masses are responsible for the greater observed gravitational fields, while for GRAS it is unknown particles whose mass dipoles are responsible for the greater observed gravitational fields. The mass dipole theory has the advantage though in that it naturally leads to the BTFR and agrees with the RAR. This is because the mass dipoles are related to baryons such that the two cannot exist independently.

GRAS was first applied to the rotation curve of the Galaxy (Penner 29a, 35) and to the rotational curves of spiral galaxies in general (Penner 30b, 35). As expected, with a theory which leads to the BTFR, the results are impressive. However, in addition to galactic rotation curves, GRAS has also been applied to the Coma cluster (Penner 31c, 35). It was found that the baryonic mass of the cluster only contributes approximately 10% of the total apparent mass of the cluster. These results were consistent with various estimates of the virial mass of the Coma cluster. Theoretical velocity dispersions for the cluster galaxies were also calculated. Comparing these results with observed velocity dispersions it was found that there was agreement if the galaxies in the Coma cluster were on near radial orbits. Given the equivalence of the field equations of GRAS and AQUAL, these results also hold for the AQUAL gravitational theory.

The theory of GRAS was also applied to binary galaxies (Penner 34). Applying the theory of GRAS to binary galaxies leads to a relationship between the line-of-sight velocity difference and the rotational velocities of the pair. The probability distribution found for this relationship is in excellent agreement with the observations taken by multiple researchers for the case of the binaries being on radial orbits. As with the Coma cluster, the binary galaxy results would hold for AQUAL too.

In addition to galaxies and systems of galaxies, GRAS has also been applied to the Solar System (Penner 36a). GRAS leads to a contribution to the gravitational field within the Solar System which only becomes significant for distances beyond a few kAU from the Sun. This is the realm of long-period comets. From a set of orbital elements for 119 long-period comets it was found that the resulting increased gravitational field greatly reduced the aphelia of these comets and increased their orbital speeds. If it is taken that the source of these comets is the Oort cloud, then it is predicted that the Oort cloud is much closer at a distance of (10 ± 2) kau and that the orbital speeds of the comets in the cloud are greater approximately 20% greater than that calculated from Newtonian gravitational theory.

GRAS also leads to an additional precession of orbiting bodies within the Solar System. It is found that the precession rate of orbiting bodies due to GRAS increases rapidly with orbiting distance. Although at present the uncertainties in observations are much greater than the predicted precession rates, it is hoped that future measurements of the orbital parameters of the most distant Solar System bodies will verify the theory. Again, these Solar System results will also apply to the AQUAL gravitational theory.

It is important to point out that while the specific function $f\left(g/g_{\mathrm{o}}\right)$ used in equation (15) has only a minor effect on the results dealing with galactic systems, as these applications are all in the weak field, within the strong gravitational field of the Solar System the specific function used is crucial. This is demonstrated by the mass dipole model put forth by Hajdukovic (16) where the function used leads to strong disagreement with observations (Banik and Kroupa 1).

In addition to the above predictions within our Solar System, an experimental test for GRAS has been proposed (Penner 2020b). This would be a direct measurement of the gravitational force between two objects carried out in a region centered on a Sun-gas planet saddle point. In these regions the gravitational field strength is comparable to the typical field values found in the outskirts of galaxies. The experiment leads to dramatic differences between Newtonian gravitational theory and what would be expected from GRAS and AQUAL.

Where GRAS and AQUAL diverge is with respect to cosmology. Measurements of CBR anisotropies by Ade et al (2014) have determined that the density parameter for baryons is 0.0482 ± 0.0016 while the Universe's overall density parameter as given by Hinshaw et al (2013) is $\Omega = 1.0027 \pm 0.0039$. In the ΛCDM model of cosmology the difference between these density parameters is due to dark matter and dark energy. Dark matter is therefore an integral part of the ΛCDM model. In GRAS, the mass dipole distribution will contribute to the energy content through equation (13). From a model of superclusters and with the baryonic density parameter set to the above determined value, the density parameter of the Universe considering just baryonic matter and the induced mass dipole contribution was found to be $\Omega = 1.08 \pm 0.19$ (Penner 33b). With GRAS dark matter and dark energy are not required. Further the dependence that the overall energy density has on the scale factor leads to a negative pressure, although the deceleration parameter is lower than what is observed and attributed to dark energy. However, this is probably due to the simplicity of the model of superclusters that is used. The key point is that an accelerating expansion falls out of the theory of gravitational anti-screening without the need of dark energy.

4. Special relativistic formulation

In the previous sections, both the AQUAL and GRAS gravitational theories were treated non-relativistically. In this section, we will provide a formulation of these theories in the context of special relativity (SR). Given the weak gravitational fields in the outer regions of galaxies taking space-time to be flat in these regions is an excellent approximation (Rowland 39, de Almeida *et al* 9). The representation of classical gravitational theory in Lorentz-invariant form is a direct analogue of the Lorentz-invariant theory of classical electromagnetism. It has been used as a steppingstone toward determining the linear GR field equations (Ohanian and Ruffini 28) and pursued as an alternate gravitational theory to GR (Fedosin 13, Nyambuya 27, Rawal and Narlikar 38). For the purposes of this manuscript the Lorentz-invariant or SR formulation of gravity will, as with Ohanian and Ruffini, provide the steps to be followed in the GR formulation undertaken in the next section. To simplify things, the fields will be considered to be static for the rest of the manuscript.

The equations of the SR or Lorentz-invariant formulation of gravity mimic those of classical electromagnetic theory. The gravitational field equation, analogous to Poisson's equation, will be given by

$$\partial_\alpha \partial^\alpha \phi^\mu = -\frac{4\pi G}{c^2} J_\mu, \tag{20}$$

where J^μ, the mass current density four-vector, is now the source of gravity. The gravitational four-potential ϕ^μ is in turn given by

$$\phi^\mu = \left(\frac{\Phi}{c}, \mathbf{A}\right), \tag{21}$$

with the gravitational scalar potential Φ now just the 0th component of the four-potential and with $\mathbf{A}$ being a three-vector analogous to the vector potential in electromagnetism. The general solution to equation (20) is

$$\phi^\mu = -\frac{G}{c^2} \int \frac{J^\mu}{|\boldsymbol{r} - \boldsymbol{r}'|} \, \mathrm{d}V'. \tag{22}$$

The mass current density four-vector is given by

$$J^\mu = \rho_o u^\mu, \tag{23}$$

where u^μ is the four-velocity and ρ_o is the total mass density in the rest frame of the given baryonic mass.

With the gauge condition

$$\partial_\alpha \phi^\alpha = 0, \tag{24}$$

the field equation, equation (20), can also be expressed as

$$\partial_\alpha C^{\alpha\mu} = -\frac{4\pi G}{c^2} J^\mu, \tag{25}$$

where the gravitational field tensor $C_{\alpha\mu}$ is given in terms of the gravitational four-potential by

$$C_{\alpha\mu} = \partial_\alpha \phi_\mu - \partial_\mu \phi_\alpha. \tag{26}$$

By equation (26), the gravitational field tensor, in the case of static fields, is given in terms of the four-potential components by

$$C_{\alpha\mu} = \begin{pmatrix} 0 & -\frac{1\partial\Phi}{c\partial \mathrm{x}} & -\frac{1\partial\Phi}{c\partial \mathrm{y}} & -\frac{1\partial\Phi}{c\partial \mathrm{z}} \\ \frac{1\partial\Phi}{c\partial \mathrm{x}} & 0 & -(\nabla\times\mathbf{A})_{\mathrm{z}} & (\nabla\times\mathbf{A})_{\mathrm{y}} \\ \frac{1\partial\Phi}{c\partial \mathrm{y}} & (\nabla\times\mathbf{A})_{\mathrm{z}} & 0 & -(\nabla\times\mathbf{A})_{\mathrm{x}} \\ \frac{1\partial\Phi}{c\partial \mathrm{z}} & -(\nabla\times\mathbf{A})_{\mathrm{y}} & (\nabla\times\mathbf{A})_{\mathrm{x}} & 0 \end{pmatrix}. \tag{27}$$

Defining the gravielectric field **g** and the gravimagnetic field **b** by

$$g = -\nabla\Phi, \tag{28a}$$

$$\mathbf{b} = \nabla\times\mathbf{A} \tag{28b}$$

then allows one to express the gravitational field tensor as

$$C_{\alpha\mu} = \begin{pmatrix} 0 & \frac{1}{c}g_{\mathrm{x}} & \frac{1}{c}g_{\mathrm{y}} & \frac{1}{c}g_{\mathrm{z}} \\ -\frac{1}{c}g_{\mathrm{x}} & 0 & -\mathrm{b_z} & \mathrm{b_y} \\ -\frac{1}{c}g_{\mathrm{y}} & \mathrm{b_z} & 0 & -\mathrm{b_x} \\ -\frac{1}{c}g_{\mathrm{z}} & -\mathrm{b_y} & \mathrm{b_x} & 0 \end{pmatrix}. \tag{29}$$

From equations (25) and (29) we can then obtain the following gravitational equivalents of Maxwell's equations,

$$\nabla\cdot\mathbf{g} = -4\pi G\rho, \tag{30a}$$

$$\nabla\times\ \mathbf{b} - \frac{1}{c^2}\frac{\partial\mathbf{g}}{\partial t} = -\frac{4\pi G}{c^2}\mathbf{J}, \tag{30b}$$

where the mass current density $J^{\mu} = (c\rho, \mathbf{J})$, with ρ being the mass density and $\mathbf{J}$ being the mass current density three-vector in the given reference frame. We can also define the following invariant scalar,

$$\omega = c\left(-C^{\alpha\mu}C_{\alpha\mu}/2\right)^{1/2} = \left(g^2 - c^2b^2\right)^{1/2}. \tag{31}$$

The preceding are the key elements in the SR or Lorentz-invariant formulation of classical gravity.

To incorporate GRAS into the above formulation, ρ_{o} in equation (23) is taken to be the sum of ρ_{bo}, the baryonic mass density in the rest frame, and ρ_{do}, the mass density in the rest frame due to the mass dipole field. Taking the four-velocities for both distributions to be the same, it follows that

$$J^{\mu} = J^{\mu}_{\mathrm{b;}} + J^{\mu}_{\mathrm{d;}}, \tag{32}$$

where $J_{\mathrm{b};}{}^{\mu}$ is the mass current density due solely to the baryonic mass and $J_{\mathrm{d};}{}^{\mu}$ is the mass current density due solely to the mass dipole field. Considering just the baryonic mass distribution leads to the following field equation, analogous to equation (25),

$$\partial_{\alpha}B^{\alpha\mu} = -\frac{4\pi G}{c^2}J_{\mathrm{b};}^{\mu}, \tag{33}$$

with the baryonic mass current density, $J_{\mathrm{b};}{}^{\mu}$, given by

$$J_{\mathrm{b};}^{\mu} = \rho_{\mathrm{bo}}u^{\mu} \tag{34a}$$

$$= (c\rho_{\mathrm{b}}, \mathbf{J_b}), \tag{34b}$$

with ρ_{b} being the baryonic mass density and $\mathbf{J_b}$ being the baryonic mass current density three-vector in the given reference frame. The gravitational tensor $B_{\alpha\mu}$ for the baryonic mass (analogous to equation (29)) can be expressed as

$$B_{\alpha\mu} = \begin{pmatrix} 0 & \frac{1}{c}g_{\mathrm{b,x}} & \frac{1}{c}g_{\mathrm{b,y}} & \frac{1}{c}g_{\mathrm{b,z}} \\ -\frac{1}{c}g_{\mathrm{b,x}} & 0 & -b_{\mathrm{b,z}} & b_{\mathrm{b,y}} \\ -\frac{1}{c}g_{\mathrm{b,y}} & b_{\mathrm{b,z}} & 0 & -b_{\mathrm{b,x}} \\ -\frac{1}{c}g_{\mathrm{b,z}} & -b_{\mathrm{b,y}} & b_{\mathrm{b,x}} & 0 \end{pmatrix}. \tag{35}$$

The next step is to determine the mass current density for the mass dipole distribution. This is a new addition to the Lorentz-invariant theory of gravity. In the model presented in section 3, it was taken that a given baryonic mass was in a sea of mass dipoles which had a mass dipole moment density distribution $\mathbf{P}$. This is analogous to the electromagnetism case of a charge embedded in a dielectric being in a sea of electric dipoles. However, just as with electromagnetism, to have a relativistic theory the field due to a moving mass dipole must also be considered. In this case a moving mass dipole will generate a gravimagnetic mass dipole moment. For example, in the case where a mass dipole with a mass dipole moment of $\mathbf{p}$ is perpendicular to its velocity $\mathbf{v}$ a gravimagnetic mass dipole moment of value

$$\mathbf{m} = \mathbf{v} \times \mathbf{p} \tag{36}$$

is created analogous to the magnetic dipole moment created by a moving electric dipole. Overall, for a given observer the mass dipole field will, in general, have both a mass dipole moment density of $\mathbf{P}$ and a gravimagnetic mass dipole moment density of $\mathbf{M}$.

Analogous to the case in electromagnetism when one is dealing with dielectrics and diamagnetic materials, we can form the following gravitational polarization tensor $D_{\alpha\mu}$ from both $\mathbf{P}$ and $\mathbf{M}$,

$$D_{\alpha\mu} = \begin{pmatrix} 0 & cP_x & cP_y & cP_z \\ -cP_x & 0 & -M_z & M_y \\ -cP_y & M_z & 0 & -M_z \\ -cP_z & -M_y & M_z & 0 \end{pmatrix}. \tag{37}$$

To obtain an understanding of this tensor, consider the case of a Lorentz transformation for two special cases. The first case is a stationary reference frame where $\mathbf{P} = P_x\hat{\boldsymbol{i}}$ and $\mathbf{M} = 0$ and we perform a Lorentz transformation to a reference frame moving at a velocity $\mathbf{v} = v\hat{\boldsymbol{i}}$.

In this new reference frame the only component found after the transformation is $P_x{}' = P_x$. This is interesting as due to length contraction one would expect $N' = \gamma N$, as the density of mass dipoles along the x-axis will increase. However, the dipoles themselves are also length contracted along the x-axis resulting in a lower dipole moment of $\mathbf{p}' = \mathbf{p}/\gamma$. The net effect is that $N'\mathbf{p}' = N\mathbf{p}$ and $P_x{}' = P_x$. The second special case is where $\mathbf{P} = P_y\,\hat{\boldsymbol{j}}$ and $\mathbf{M} = 0$ and we undergo the same transformation along the x-axis. In this case, in the new reference frame we end up with $P_y{}' = \gamma P_y$ as expected as $N' = \gamma N$ and there is no length contraction for the individual mass dipoles. However, in the new reference frame the transformation also leads to $M_z{}' = -\gamma v P_y$ which makes sense as $N' = \gamma N$ and by (36) for this case $\mathbf{m}' = -vp\hat{\boldsymbol{k}}$.

Given the gravitational-polarization tensor $D_{\alpha\mu}$, the resulting mass current density four-vector $J_{\mathrm{d};\mu}$ resulting from the mass dipole distribution is determined by

$$\partial_\alpha D^{\alpha\mu} = -J^{\mu}_{\mathrm{d};}, \tag{38}$$

$$J^{\mu}_{\mathrm{d};} = \rho_{\mathrm{do}} u^{\mu}. \tag{39}$$

Expressing equation (38) explicitly in terms of $\mathbf{P}$ and $\mathbf{M}$ leads to

$$\nabla \cdot \mathbf{P} = -\rho_{\mathrm{d}} \tag{40a}$$

and

$$(\nabla \times \mathbf{M}) - \frac{\partial \mathbf{P}}{\partial t} = -J_{\mathrm{d};}, \tag{40b}$$

with the mass current density for the dipole field given by $J^{\mu}_{\mathrm{d};} = (c\rho_{\mathrm{d}}, \mathbf{J_d})$, where ρ_{d} is the mass density and $\mathbf{J_d}$ is the mass current density three-vector in the given reference frame. This again is analogous to Maxwell's equations for electromagnetism in the case of the bound charge and current densities in dielectrics and diamagnetic materials.

We are now in a position to determine the SR field equation for both the GRAS and AQUAL gravitational theories. Equations (23), (25), (33), and (38) lead to

$$\frac{c^2}{4\pi G}\partial_\alpha C^{\alpha\mu} = \frac{c^2}{4\pi G}\partial_\alpha B^{\alpha\mu} + \partial_\alpha D^{\alpha\mu}. \tag{41}$$

The key step in the derivation is that by comparing the equations given by (30) with (40) one is led to the following relativistic equivalent of (15):

$$D^{\alpha\mu} = \frac{c^2}{4\pi G} f(\omega/g_{\mathrm{o}}) C^{\alpha\mu}, \tag{42}$$

with f now being a function of the invariant scalar ω. Substituting (42) into (41) and rearranging then results in

$$\partial_\alpha((1-f(\omega/g_o))C^{\alpha\mu}) = \partial_\alpha B^{\alpha\mu}, \tag{43}$$

which by equation (33) then leads to

$$\partial_\alpha((1-f(\omega/g_{\mathrm{o}}))C^{\alpha\mu}) = -\frac{4\pi G}{c^2} J^{\mu}_{\mathrm{b};}. \tag{44}$$

Equation (44) is the SR field equation for both the GRAS and AQUAL gravitational theories, where in the case of AQUAL $1-f(\omega/g_{\mathrm{o}})$ would be replaced by $\mu(\omega/g_{\mathrm{o}})$. As is seen with (44) the function f, and therefore μ, now depends on ω, which is a combination of both g and the gravimagnetic field b. In the nonrelativistic limit $\omega \rightarrow g$.

Taking the 0th component of equation (44) along with the nonrelativistic limit where $\omega \rightarrow g$ and $\rho_\mathrm{b} \rightarrow \rho_\mathrm{bo}$, results in

$$\partial_\alpha((1 - f(\omega/g_\mathrm{o}))C^{\alpha 0}) = -\frac{4\pi G}{c^2} J_{\mathrm{b};}{}^{0} = -\frac{4\pi G}{c}\rho_\mathrm{bo}, \tag{45}$$

which by equation (29) leads to the nonrelativistic GRAS and AQUAL field equation

$$\nabla \cdot \nabla\Phi = 4\pi G\rho_\mathrm{bo} + \nabla \cdot \left(f\left(g/g_\mathrm{o}\right)\nabla\Phi\right). \tag{46}$$

The starting point of this derivation of the relativistic field equation was the general field equation as given by equation (20). This equation falls out of Euler–Lagrange equation with the Lagrangian given by

$$\mathcal{L} = -\frac{c^2}{16\pi G} C_{\mu\nu}C^{\mu\nu} + J^\mu \phi_\mu. \tag{47}$$

This Lagrangian is the relativistic version of the Newtonian Lagrangian as given by (4) which can be seen by substituting equation (31) into equation (47) leading to

$$\mathcal{L} = \frac{1}{8\pi G}\omega^2 + J^\mu \phi_\mu. \tag{48}$$

The Lagrangian for an object of mass m moving in the above gravitational field is given by

$$\mathrm{L} = \frac{1}{2}\mathrm{m}\ \ \mathrm{u}^\mu \mathrm{u}_\mu + \mathrm{m}\ \ \mathrm{u}^\mu \mathrm{A}_\mu. \tag{49}$$

Substituting equation (49) into the Euler–Lagrange equation then leads to the following equation of motion

$$\frac{\mathrm{d}u_\mu}{\mathrm{d}\tau} = C_{\mu\alpha}\,\mathrm{u}^\alpha, \tag{50}$$

where u^α is the four-velocity of the object. In terms of $\mathbf{g}$ and $\mathbf{b}$, equation (50) becomes

$$\frac{\mathrm{d}\left(\gamma\mathbf{v}\right)}{\mathrm{d}t} = \mathbf{g} + \mathbf{v} \times \mathbf{b}, \tag{51}$$

analogous to the relativistic Lorentz force law in electromagnetism.

As has been shown, the mass dipole theory fits quite naturally with SR and leads to a relativistic field equation both for itself and for AQUAL. The preceding SR steps that were taken will now be followed to derive the field equation in the context of general relativity.

5. General relativistic formulization

For the weak gravitational fields found in the outer regions of galaxies the linear approximation of GR is suitable. In this case, the general linear GR field equation, as given by Ohanian and Ruffini (28), is

$$\partial_\alpha\partial^\alpha h^{\mu\nu} + \partial^\mu\partial^\nu h - (\partial_\alpha\partial^\nu h^{\mu\alpha} + \partial_\alpha\partial^\mu h^{\nu\alpha}) - \eta^{\mu\nu}\partial_\alpha\partial^\alpha h + \eta^{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta} = -\frac{4\pi G}{c^4}T^{\mu\nu}, \tag{52}$$

where $h^{\mu\nu}$ is the gravitational potential tensor and $\mathrm{T}^{\mu\nu}$, the source of gravity in GR, is the energy momentum tensor. Under the gauge condition

$$\partial_\mu \left(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h \right) = 0 \tag{53}$$

and with the substitution

$$\varphi^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h, \tag{54}$$

this field equation greatly simplifies to

$$\partial_\alpha \partial^\alpha \varphi^{\mu\nu} = -\frac{4\pi G}{c^4} T^{\mu\nu}, \tag{55}$$

with the gauge condition now becoming

$$\partial_\mu \varphi^{\mu\nu} = 0. \tag{56}$$

The solution to equation (55) is

$$\varphi^{\mu\nu} = -\frac{G}{c^4} \int \frac{T^{\mu\nu}}{|\boldsymbol{r} - \boldsymbol{r}'|}\, \mathrm{d}V' \tag{57}$$

with the gravitational scalar potential now just the 00 component of the gravitational potential tensor $\varphi^{\mu\nu}$, i.e.

$$\varphi^{00} = \frac{\Phi}{c^2}. \tag{58}$$

Under the linear approximation, both the self-energy of the gravitational field and the interaction energy between the mass distribution and the field are neglected. In this case the energy–momentum tensor is given by

$$T^{\mu\nu} = \rho_\mathrm{o} u^\mu u^\nu, \tag{59}$$

with the total mass density ρ_o being in the case of GRAS equal to the sum of ρ_bo and ρ_do. With the four-velocities for both mass density distributions taken to be the same, the total energy–momentum tensor can be expressed as

$$T^{\mu\nu} = T_{\mathrm{b};}^{\mu\nu} + T_{\mathrm{d};}^{\mu\nu}, \tag{60}$$

where $T_{\mathrm{b};}{}^{\mu\nu}$ is the energy–momentum tensor solely for the baryonic mass and $T_{\mathrm{d};}{}^{\mu\nu}$ is the energy–momentum tensor solely for the mass dipole distribution.

Analogously to equation (25), the field equation as given by equation (55), under the given gauge condition, can also be expressed as

$$\partial_\alpha \Psi^{\alpha\mu\nu} = \frac{4\pi G}{c^4} T^{\mu\nu}, \tag{61}$$

where the tensor $\Psi_{\alpha\mu\nu}$ is given by

$$\Psi_{\alpha\mu\nu} = \partial_\nu \varphi_{\alpha\mu} + \partial_\mu \varphi_{\nu\alpha} - \partial_\alpha \varphi_{\mu\nu}. \tag{62}$$

We can also define the following invariant scalar:

$$\Omega = c^2\left(-\Psi^{\alpha\mu\nu}\Psi_{\alpha\mu\nu}/2\right)^{1/2}. \tag{63}$$

Proceeding now as with the SR formulation, we first consider the field equation for just the baryonic mass distribution, which in this case will be

$$\partial_\alpha \Psi_{\mathrm{b};}^{\alpha\mu\nu} = \frac{4\pi G}{c^4} T_{\mathrm{b};}^{\mu\nu}, \tag{64}$$

where the tensor ${\Psi_{\mathrm{b};}}^{\alpha\mu\nu}$ is given by

$$\Psi_{\mathrm{b};\alpha\mu\nu} = \partial_\nu \varphi_{\mathrm{b};\alpha\mu} + \partial_\mu \varphi_{\mathrm{b};\nu\alpha} - \partial_\alpha \varphi_{\mathrm{b};\mu\nu} \tag{65}$$

and the energy–momentum tensor for the baryonic mass distribution is given by

$$T_{\mathrm{b};}^{\mu\nu} = \rho_{\mathrm{bo}} u^\mu u^\nu \tag{66a}$$

$$= {J_{\mathrm{b};}}^{\mu} u^\nu. \tag{66b}$$

The field equation due solely to the mass dipole distribution will in turn be given by

$$\partial_\alpha \Psi_{\mathrm{d};}^{\alpha\mu\nu} = T_{\mathrm{d};}^{\mu\nu}, \tag{67}$$

with the tensor ${\Psi_{\mathrm{d};}}^{\alpha\mu\nu}$ given by

$$\Psi_{\mathrm{d};\alpha\mu\nu} = \partial_\nu \varphi_{\mathrm{d};\alpha\mu} + \partial_\mu \varphi_{\mathrm{d};\nu\alpha} - \partial_\alpha \varphi_{\mathrm{d};\mu\nu} \tag{68}$$

and the energy–momentum tensor for the dipole field given by

$$T_{\mathrm{d};}^{\mu\nu} = \rho_{\mathrm{do}} u^\mu u^\nu \tag{69a}$$

$$= J_{\mathrm{d};}^{\mu} u^\nu \tag{69b}$$

$$= -u^\nu \partial_\alpha D^{\alpha\mu}. \tag{69c}$$

We can now determine the linear GR field equation for both the GRAS and AQUAL gravitational theories. By equations (60), (61), (64), and (67), we have

$$\frac{c^4}{4\pi G}\partial_\alpha \Psi^{\alpha\mu\nu} = \frac{c^2}{4\pi G}\partial_\alpha \Psi_{\mathrm{b};}^{\alpha\mu\nu} + \partial_\alpha \Psi_{\mathrm{d};}^{\alpha\mu\nu}. \tag{70}$$

The key step is that the general relativistic equivalent of equation (42) will be given by

$$\Psi_{\mathrm{d};}^{\alpha\mu\nu} = \frac{c^4}{4\pi G} f(\Omega/g_{\mathrm{o}})\Psi^{\alpha\mu\nu}, \tag{71}$$

with f now being a function of the scalar Ω. Substituting equation (71) into equation (70) then leads to

$$\partial_\alpha((1-f(\Omega/g_{\mathrm{o}}))\Psi^{\alpha\mu\nu}) = \partial_\alpha \Psi_{\mathrm{b};}^{\alpha\mu\nu}, \tag{72}$$

which by equation (64) leads to

$$\partial_\alpha((1-f(\Omega/g))\Psi^{\alpha\mu\nu}) = \frac{4\pi G}{c} T_{\mathrm{b};}^{\mu\nu}. \tag{73}$$

Equation (73) is the linear GR field equation for both the GRAS and AQUAL gravitational theories, where in the case of AQUAL $1 - f(\Omega/g_o)$ would be replaced by $\mu(\Omega/g_o)$.

Taking the 0th component of equation (73) along with the following equalities

$$\Psi^{\alpha\mu 0} = -\frac{1}{c}C^{\alpha\mu}, \tag{74a}$$

$$\varphi^{\mu 0} = \frac{1}{c}\phi^{\mu}, \tag{74b}$$

$$\Omega_{\nu \rightarrow 0} = c^2\left(-\Psi^{\alpha\mu 0}\Psi_{\alpha\mu 0}/2\right)^{1/2} = c\left(-C^{\alpha\mu}C_{\alpha\mu}/2\right)^{1/2} = \omega, \tag{74c}$$

and

$$T_{\text{b;}}^{\mu 0} = c\gamma\ J_{\text{b;}}^{\mu} \tag{74d}$$

results in

$$\partial_{\alpha}((1 - f(\omega/g_o))C^{\alpha\mu}) = -\frac{4\pi G}{c^2}\gamma\ J_{\text{b;}}^{\mu}. \tag{75}$$

In the limit where terms quadratic in velocities are neglected, $\gamma \rightarrow 1$ and equation (75) becomes the SR field equation (44). Of course, equation (75) is just the 0th component of the linear GR field equation (73).

While the preceding is based on a linear approximation of GR, it does demonstrate how GRAS and AQUAL can be handled in GR. The key is that the gravitational contribution from the dipole field, $T_{\text{d;}}{}^{\mu\nu}$, is no different than a contribution from a baryonic mass distribution, $T_{\text{b;}}{}^{\mu\nu}$. Therefore, with respect to this, GR holds and no new relativistic theory such as TeVeS is needed.

6. Conclusion

As demonstrated both the GRAS and AQUAL gravitational theories lead to the same field equation. In the case of AQUAL, a new gravitational theory is needed. In the case of GRAS, Newtonian gravitational theory holds but with an added gravitational source. Not only does the mass dipole theory provide a physical mechanism behind the BTFR and RAR, it also fits naturally with SR and linear GR. The relativistic field equations derived hold for both GRAS and AQUAL gravitational theories.

The mass dipole theory provided is general in that it does not depend on the specific mass dipole model. Of course, the Achilles heel of the mass dipole theory are the mass dipoles themselves. Mass dipoles or individual particles with mass dipole moments are not known to exist. Of course, given that the only impact of a particle having a mass dipole moment will be with respect to its gravitational field, there is no experimental evidence for or against the proposition. A discussion of the different proposed mass dipoles, including the author's own, and how they fit in with current physical theory is included in Penner (36a). As discussed there, these proposals do have issues and certainly a proper theory for mass dipoles is required. Such a theory should lead to a specific interpolating function as the standard interpolating function given by equation (7) is solely based on observations and is only one of many possibilities. This is a major benefit of having a physical theory underlying the BTFR and RAR, as deriving a theoretically based interpolating function is possible. With such a theory, it is the author's belief that issues such as requiring a dark matter component in addition to AQUAL or GRAS like theories to explain certain observations will be resolved. However, even currently without such a theory, GRAS is a viable alternative to the theory of dark matter.

Data availability statement

No new data were created or analysed in this study.

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